Chapter 2: Beam Dynamics

2.1 Conventions

We use right-handed (x, y, z) coordinates for a linear transport system in which x is positive in the vertically upward direction and z is positive in the direction of the beam. We will use nonrelativistic formulae, relativistic corrections being negligible throughout this work. There is often confusion of the relativistic "beta", the ratio of a velocity to the speed of light, with the "beta function" of accelerator physics, to be defined shortly. We will denote the relativistic quantity by $\beta_r = v/c$, all other occurrences of β being the accelerator function, which depends on the distance along the focusing channel. All external focusing fields will be assumed linear in transverse displacement, with a given periodicity in z. We will denote derivatives with respect to z by primes. Thus, for example, $x' \equiv \partial x/\partial z$. Finally, we will denote the RMS values of a parameter, such as the beam offset in the x dimension, by a tilde, as

$$\tilde{x} \equiv \sqrt{\langle (x - \overline{x})^2 \rangle}.$$

Space-charge will be assumed to be the only source of non-linear fields, and for most of this work only the linear part of this field is considered. We will assume a monoenergetic longitudinal beam distribution without acceleration. The experimental apparatus incorporates electrostatic quadrupoles for beam focusing, and we will write all focusing fields as electric fields, using mks units. Recall that the magnetic equivalent involves substitution of $\underline{\mathbf{v}} \times \underline{\mathbf{B}}$ for $\underline{\mathbf{E}}$. The space-charge field will be calculated locally as if it were purely transverse, that is, neglecting beam envelope variations.

Periodic focusing systems are well-covered in the classic paper by Courant and Snyder [26], which includes the limitations placed on circular machines by lens errors. A very comprehensive treatment of accelerators with good mathematical background material is given by Brück [27]. Some additional material on space-charge-dominated transport is included in Appendix A.

2.2 Envelope Descriptions

2.2.1 Beam with negligible space-charge

In a periodic focusing channel with a restoring force linear in the transverse displacements x and y, the equations of motion for a particle may be written (neglecting self-fields) as

$$\begin{aligned} x''(z) + K_x(z)x(z) &= 0 \\ y''(z) + K_y(z)y(z) &= 0, \end{aligned} \tag{2.1}$$

where K_x and K_y are periodic functions of z. For continuous solenoid focusing $K_x(z) = K_y(z) =$ constant.; for an A.G. system $K_x(z) = -K_y(z) = K(z)$. Unless otherwise stated, we will assume henceforth that the focusing is from a quadrupole array, so that the force constants in the x and y planes differ only in sign. We will assume that the focusing array has the focus-drift-defocus-drift (FODO) geometry, in which the lens fields reverse sign with each lens along the array.

From Eqns. 2.1 one obtains the equations for the beam envelopes $a(z)$ and $b(z)$ in the x and y planes, respectively, (as in Lawson [28], for example)

$$\begin{aligned} a''(z) + K(z)a(z) - \frac{\epsilon_x^2}{a^3(z)} &= 0 \\ b''(z) - K(z)b(z) - \frac{\epsilon_y^2}{b^3(z)} &= 0, \end{aligned} \tag{2.2}$$

where ϵ_x and ϵ_y are the "emittances" for the two planes, defined below.

In an electrostatic system, the focusing field coefficient $K(z)$ is given by

$$K(z) = \frac{q}{mv_z^2}\frac{\partial \mathrm{E}_x}{\partial x},$$

which alternates in sign from lens to lens as E_x changes sign. Here m is the particle mass, q its charge, and v_z the z-velocity. It is conventional in the field of accelerator physics to call the (x, x') space "phase space," even though in mechanics that term is usually reserved for the space described by the canonically conjugate variables (x, p_x). (We will neglect all vector potential effects, so that p_x is purely the mechanical momentum.) The area occupied by the beam in (x, x') space is π times the product of the semi-axes of the ellipse. The product of the semi-axes of the ellipse is denoted by ϵ, and so the area of the beam in phase space is $\pi\epsilon$. When quoted in the (scaled) canonical phase space $(x, p_x/mc)$, the area is $\pi\epsilon_{\mathrm{N}}$. The quantity ϵ is called the "emittance," or "unnormalized emittance," and is the quantity occurring in the envelope equations. The quantity $\epsilon_{\mathrm{N}} = \epsilon\beta_r\gamma$, where β_r and γ are the usual relativistic factors, is called the "normalized emittance." It is particularly useful because it remains constant upon acceleration of the beam, in the absence of nonlinear forces.

The "acceptance" is the transverse phase space area into which particles may be injected without subsequent loss to the walls. A beam is called "matched" in a periodic lattice if its envelope has the same periodicity as the lattice. The envelope of a mismatched beam undergoes oscillations about the matched solution, requiring a larger aperture for the same beam emittance.

The motion of individual particles in the x plane can be written

$$x(z) = \sqrt{\beta(z)C}\ \sin\{\psi(z) + \psi_0\}, \tag{2.3}$$

where $\psi(z)$ is called the betatron phase function, $\beta(z)$ is called the envelope function, and C and ψ_0 are constants depending upon the initial conditions. The particle motion is broadly sinusoidal ("betatron oscillations") with a superposed higher frequency flutter component described by $\sqrt{\beta}$. The flutter occurs as a particle is alternately focused and defocused by the lenses. The average betatron motion is due to the average restoring force of the A.G. channel.

For any given beam particle undergoing linear focusing, there is a constant of the motion,

$$\left(\frac{x}{\beta}\right)^2 + \left(x' - \frac{\beta'}{2\beta}x\right)^2 = C, \tag{2.4}$$

where β is the envelope function defined above. For the outermost particles in phase space of a matched beam, $C = \epsilon$. The largest offset in x of any particle is given by $\sqrt{\beta\epsilon}$, so this quantity gives the radius of the beam. The particle oscillation frequency in linear accelerators is usually characterized by σ_0 in units of degrees of phase advance per period (analogous to a wave-number), often loosely called the zero-current "tune" of the lattice.

The individual particle motion we have been describing to this point is referred to as "incoherent." In addition one can have a "coherent" motion of the beam about the axis if the beam is misaligned. In this case the centroid of the beam oscillates about the axis with the betatron wave-number σ_0, as if the beam were a macroparticle.

2.2.2 Beam with linear space-charge field included

In the case of azimuthally symmetric focusing, uniform in z, many distributions which include self-fields in a self-consistent way may be written either explicitly or implicitly [29], because the total energy is a constant of the motion. However, only one of these, called the "Kapchinskij-Vladimirskij" or "K-V" distribution [30], is known to be generalizable to the case of periodic focusing, for which the transverse Hamiltonian is not a constant of the motion. The K-V distribution is a microcanonical distribution in terms of a modified transverse Hamiltonian, based on the constant of the motion given in Eqn. 2.4. That is, all the beam particles lie on a single surface in the total transverse phase space (x, x', y, y'). This particular distribution function results in a uniform particle density within an elliptical boundary in (x, y) space. This gives rise to linear space-charge fields, satisfying the requirements for Eqn. 2.4 to give a constant of the motion. The envelope equations for the two transverse planes are coupled by the self-field, although the emittances ϵ_x and ϵ_y ideally are independent constants. Denoting the beam radius in the x plane by $a(z)$ and in the y plane by $b(z)$, we add the defocusing space-charge term to Eqns. 2.2 and obtain the K-V envelope equations

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$$\begin{aligned} a''(z) + K(z)a(z) - \frac{2Q}{a(z)+b(z)} - \frac{\epsilon_x^2}{a^3(z)} &= 0 \\ b''(z) - K(z)b(z) - \frac{2Q}{a(z)+b(z)} - \frac{\epsilon_y^2}{b^3(z)} &= 0, \end{aligned} \tag{2.5}$$

where $Q = \frac{\cancel{2}\,qI}{2\pi\epsilon_0 m v_z^3}$ in the nonrelativistic limit. We calculate the matched envelope parameters for our experiment by direct integration of these equations, using the focusing field representation given in Appendix D.

Within this linear field model, a useful relation between the values of ϵ, σ_0, and the space-charge depressed "tune", σ, may be found for a uniform focusing channel. In this case, $a = b$ and K is constant with no sign difference between the dimensions, and we have

$$a''(z) + Ka(z) - \frac{Q}{2a(z)} - \frac{\epsilon^2}{a^3(z)} = 0. \tag{2.6}$$

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The matched beam envelope in a uniform focusing channel is constant, which we will denote by R. The trajectory equation for the particles in the matched beam is

$$x''(z) + Kx(z) - \frac{Q}{2R^2}x(z) = 0.$$

We see that the zero-current wave number κ_0 for the particle oscillation is given by $\kappa_0^2 = K$, and the space-charge depressed value κ is given by

$$\kappa^2 = \kappa_0^2 - Q/2R^2. \tag{2.7}$$

If we further note that the value for ϵ is given by the product of the beam radius and the maximum crossing angle of particles at the beam axis, $\epsilon = R^2\kappa$, then we may obtain the relation

$$\kappa^2 + \frac{Q}{2\epsilon}\kappa - \kappa_0^2 = 0.$$

We use this result to obtain the proportionality (written in the periodic channel notation given above)

$$\sigma \propto \frac{\epsilon}{I}(\sigma_0^2 - \sigma^2).$$

This is approximately valid for $\sigma_0 < 90°$, and implies that if σ much less than σ_0, then $\sigma \propto (\epsilon/I)\sigma_0^2$. If σ_0 and the current are constant, then σ increases monotonically with ϵ.

By perturbing Eqn. 2.6, we may calculate the frequency of envelope oscillations for the envelope mode in which the two dimensions oscillate in-phase. Denoting the perturbation by δ, we obtain

$$\delta'' + \left(K + \frac{Q}{2R^2} + 3\frac{\epsilon^2}{R^4}\right)\delta = 0. \tag{2.8}$$

We eliminate ϵ in favor of the matched beam radius R by using Eqn. 2.6. By using the expressions for κ_0 and κ above we obtain the envelope frequency k, using the periodic channel notation,

$$k^2 = 2\sigma_0^2 + 2\sigma^2. \tag{2.9}$$

This mode frequency will be used in Ch. 5. In the zero-current limit, $\sigma \to \sigma_0$, and so $k \to 2\sigma_0$. This is because when the particles have executed half of a betatron oscillation, the envelope has executed one full oscillation. The high-current envelope oscillation frequency could have been written in terms of the plasma frequency, showing explicitly the space-charge-dominated nature of the mode in that limit. We have used the channel strength parameter σ_0, linearly related to the cold-beam plasma frequency, as may be seen from Eqn. 2.7 by rewriting Q/R^2 in terms of ω_p^2.

2.2.3 RMS envelope description including nonlinear space-charge

An alternative envelope analysis has been given by Sacherer [31] and Lapostolle [14] in terms of various moments of the trajectory equation, averaged over an arbitrary beam distribution. The resulting hierarchy of coupled moment equations is examined for a low-order quantity which can be approximated from other considerations to close the chain of variables and equations. The unnormalized RMS emittance is such a quantity, formally defined by

$$\epsilon_{\mathrm{RMS}} \equiv \sqrt{\langle(x-\overline{x})^2\rangle\langle(x'-\overline{x'})^2\rangle - \langle(x-\overline{x})(x'-\overline{x'})\rangle^2}, \tag{2.10}$$

where both the brackets $\langle\rangle$ and overlines denote an average over the distribution. The nonlinear portion of the total field drives the z variation of the RMS emittance. The equations for the RMS radius of the beam in terms of the linearized self-field and the RMS emittance are identical in form to the K-V equations (Eqns. 2.5) with the restriction that the beam distribution in (x, y) space must have elliptical symmetry. The RMS emittance is either taken to be constant or is approximated in other simple ways from known behavior. For any beam with constant RMS emittance, the RMS beam envelope is well-modeled by the envelope equations. For a K-V distribution the usual radius is equal to $2\tilde{x}$, where $\tilde{x}$ is the RMS beam radius, and the usual emittance, ϵ, is equal to $4\epsilon_{\text{RMS}}$. We will identify $2\tilde{x}$ with the beam radius calculated from the envelope equations (Eqns. 2.5), and will use $4\epsilon_{\text{RMS}}$ for the beam emittance.

2.3 Bore Requirements as a Function of Lens Strength

A desirable property of an accelerator for many applications, including HIF, is that the average current density over the bore of the accelerator be high. For a single beam of fixed current, this primarily involves minimizing the maximum beam radius, although for multiple beams the packing fraction for the beams within the bore is also important.The relationship between lattice strength and envelope size differs considerably between emittance-dominated and current-dominated beam transport, as we will now illustrate using the thin-lens quadrupole lattice. Because of the symmetry, the maximum in the beam radius occurs in the focusing lens. For a thin-lens FODO lattice with focusing period $2L$, lens focal length $\pm f$, and no space-charge, we obtain

$$\beta_{\text{max}} = 2L\frac{1+\sin(\sigma_0/2)}{\sin\sigma_0}, \tag{2.11}$$

where $\sin(\sigma_0/2) = L/2f$. Note the divergence of β_{max} as σ_0 approaches 180°, for which the particle becomes resonant with the focusing lattice. For given values of ϵ and L, the required beam aperture is minimized (for the thin lens model) for $\sigma_0 \simeq 76.3°$.

In contrast, for an ideal zero-emittance beam with non-zero current, the minimum in required aperture occurs at a lens strength well beyond the $\sigma_0 =$ 180° limit for zero-current beams (see Appendix A). The beam itself, viewed as a macroparticle, will become resonant with the focusing if σ_0 is raised too high, even though the individual particles experience a much weaker overall focusing and do not become resonant with the external focusing as in the zero-current case. This provides a great incentive to determine the strongest lattice usable for high current transport. For low emittance, in the absence of collective instability, and with perfect alignment of the lattice and beam, very high intensity beams could be transported. The different response of the envelope to lens strength for emittance-dominated and for current-dominated beams is due to the much different dependence on the beam radius of the space-charge defocusing term and the emittance term in the envelope equations, Eqns. 2.5.